Half-checking propagators

Mikael Zayenz Lagerkvist[1][0000−0003−2451−4834] and
Magnus Rattfeldt[2][0000−0001−5036−3107]

[1] research@zayenz.se, https://zayenz.se
[2] research@rattfeldt.se

Abstract. Central to the success of constraint programming are propagators, contracting functions removing values proven not to be in any solution of a given constraint. The literature contains numerous propagation algorithms, for many different constraints, and common to all these propagation algorithms is the notion of correctness: only values that appear in *no solution* to the respective constraint may be removed. In this paper *half-checking propagators* are introduced, for which the only requirements are that identified solutions (by the propagators) are actual solutions (to the corresponding constraints), and that the propagators are contracting. In particular, a half-checking propagator may *remove solutions* resulting in an incomplete solving process, but with the upside that (good) solutions may be found faster. Overall completeness can be obtained by running half-checking propagators as one component in a portfolio solving process. Half-checking propagators opens up a wider variety of techniques to be used when designing propagation algorithms, compared to what is currently available.
A formal model for half-checking propagators is given with a detailed description of how to support such propagators in a system. Three general ideas for creating half-checking propagators are introduced, each with an example half-checking propagator for the `cost-circuit` constraint. The new propagators are implemented and tested in the Gecode system.

1 Introduction

Constraint programming has been successful in a wide variety of settings, and central to the success of constraint programming is the multitude of smart and efficient propagation algorithms devised. Propagation is all about removing values that are not in any solution to a constraint, and it is what separates constraint programming from generate-and-test. In constraint programming, we are justifiably proud of being able to effectively combine algorithms from many different fields implemented as propagators, so that a model effortlessly and without fear of adverse interactions can use intelligent scheduling algorithms for `disjunctive` and `cumulative` such as not-first/not-last and energetic reasoning, dynamic programming algorithms for `regular`, `bin-packing`, and `knapsack`, maximum flow reasoning for `global-cardinality`, and arithmetic and Boolean reasoning.

Unfortunately, designing good propagation algorithms is hard. It is hard not only since the specific problems they model are hard, but they are hard for a more fundamental reason. Propagators are required to be *correct*; a propagator must never remove a value from a variable that may still be a solution to the constraint. This means that propagation is not actually concerned with finding a solution but about proving that no solution exists for a certain variable-value pair, and that is a subjectively harder problem. The requirement for correctness also means that there is an upper limit on the amount of propagation that can be done, and this limit (*domain consistency* [38]) is often the ultimate goal when designing a new propagator. Unfortunately, even if a propagator is domain consistent it does not mean that it performs a high amount of propagation: perhaps all values can still be part of some solution for the constraint.

In this paper we propose a new type of propagators, that we call *half-checking propagators*. By relaxing the requirements of propagators to a bare minimum for ensuring soundness (found solutions must be constraint solutions), we open up for a wider variety of techniques that may be used when designing propagation algorithms. On the downside, such propagators are no longer correct, which means that the overarching solving process is no longer complete. On the upside, however, such propagators can deploy new and stronger reasoning (possibly even stronger than domain consistency), with the hope that the search is then guided towards promising parts of the search space.

Local search, heuristics, and approximation algorithms are all examples of incomplete methods, methods that are often geared towards finding good enough solutions quickly, not a provably optimal solution. In constraint programming, the most well known and successful incomplete technique is Large Neighborhood Search [53]. In contrast to LNS, we embrace the incompleteness earlier by lifting it into the propagators, the heart of a constraint programming solver. Similar to all incomplete strategies, completeness can be regained by combining with a complete solution method in a portfolio solver.

Contributions. This paper introduces the novel concept of *half-checking propagators*, including a formal model, a full exploration on how to integrate into a realistic system, and how to use in a portfolio solver. Three general techniques for designing half-checking propagators are defined. For all three, an example propagator using the technique is developed for the `cost-circuit` constraint. An implementation in the Gecode constraint programming system has been made.

2 Constraint programming

In order to be clear about the specifics, a formal model of constraint programming is needed, as is knowing the standard requirements on propagators.

2.1 Constraint satisfaction problems

Let $\mathcal{P}(s)$ be the *power-set* of s, that is the set of all subsets of s. The set of all functions from the set A to the set B is denoted $A \to B$. Let $\lambda x.E$ be the function from the argument x to the expression E.

A constraint satisfaction problem is defined over a finite set of *variables* $Var = \{x_1, \ldots, x_n\}$ and a finite set of *values* Val. An *assignment* $a \in Asn$ maps each variable in Var to a value in Val, $Asn = Var \rightarrow Val$. For a set of variables $x \subseteq Var$, Asn_x is the assignments where the arguments are restricted to x, and a_x is similarly an assignment restricted to x. A *constraint* $c \in Con$ over variables $var(c) \subseteq Var$ is defined as the set of assignments that are solutions to that constraint: $Con = \cup_{x \subseteq Var} \mathcal{P}(Asn_x)$. When necessary and without loss of generality, any constraint is extended to all variables Var by allowing all combinations of values for the added variables, for all solutions.

A *domain* $d \in Dom$ maps each variable to a subset of the values, $Dom = Var \rightarrow \mathcal{P}(Val)$. For simplicity, all domains where at least one variable is mapped to the empty set are equated and represented by the fully empty domain ($\perp = \lambda x.\{\}$). A domain d induces a set of assignments ($asn(d) = \{a \mid \forall x.\, a(x) \in d(x)\}$), and can thus be considered as a constraint. Set operations and relations are lifted to domains, constraints and assignments point-wise over the variables.

The domain of a constraint is defined as $dom(c) = \lambda x.\{v \mid \exists a \in c.\, a(x) = v\}$. Note that the domain of a constraint in turn induces a much weaker constraint than the original. For example, the equality constraint eq for two variables contains just $|Val|$ assignments, while $asn(dom(eq))$ contains all $|Val|^2$ assignments.

A constraint satisfaction problem (CSP) is a tuple $\langle d, C\rangle$ of a domain d and a set of constraints C. An assignment a is a *solution* to a CSP iff $a \in d$ and $\forall c \in C.\, a \in c$. The set of all solutions to a CSP csp is given by the function $sol(csp)$. A function $solve \in CSP \rightarrow \mathcal{P}(Asn)$ finds solutions for a CSP. Such a function is *sound* iff $solve(csp) \subseteq sol(csp)$ (all solutions found are actually solution). It is *complete* iff $solve(csp) = sol(csp)$ (solving finds all solutions).

2.2 Propagators and models

A *propagator* p for a constraint c is a function[3] from domains to domains ($p \in Dom \rightarrow Dom$), with the following properties.

Contracting $\forall d \in Dom$, $p(d) \subseteq d$ must hold.
Local $\forall d \in Dom$, if $x \notin var(c)$, then $p(d)(x) = d(x)$.
Checking $\forall a \in Asn$, $p(dom(\{a\})) = dom(\{a\})$ iff $a \in c$.
Weakly monotonic $\forall d \in Dom$ and assignments $a \in d$, $p(dom(\{a\})) \subseteq p(d)$.

Contracting means that a propagator only removes values from domains, never adds values. *Local* means that a propagator only removes values from the variables involved in the constraint. *Checking* means that a propagator recognizes all solutions to a constraint since no values are removed for those assignments. *Weakly monotonic* means that if an assignment is a fix-point of a propagator (and thus a solution to the constraint), then the propagator does not remove that assignment from a domain it is in. Any propagator that is weakly monotonic and checking is *correct* for its constraint [49], with correct defined as follows.

[3] As remarked in [49], propagators do not need to be functions, and can be arbitrary relations in $Dom \times Dom$, e.g., as a model for randomized propagation. For ease of explanation and notation, we use functions and leave the generalization unstated.

Definition 1 (Correct). *A propagator p is correct for constraint c, iff*

$$\forall a \in c. \forall d \in Dom. a \in asn(d) \implies a \in asn(p(d))$$

Let the constraint of a propagator p be referred to as c_p. A *constraint model* is a combination of a domain and a set of propagators $\langle d, P\rangle$. This is very similar to a CSP as defined above, and a model can be transformed to a CSP using $csp(\langle d, P\rangle) = \langle d, \{c_p | \forall p \in P\}\rangle$. A CSP defines the semantics of a problem using an extensional specification, while a model is an intensional specification geared towards computing solutions to the CSP.

Solving a model is usually done by interleaving fix-point computation of the propagators with search using heuristic decomposition of the model (branching or labeling). We leave the details of solving opaque, assuming a function $solve(\langle d, P\rangle)$ that returns all solutions that are fix-points of all propagators.

In [49], Schulte and Tack introduced weak monotonicity and showed that the above properties for propagators[4] are necessary and sufficient to get *sound* and *complete* solving when combined with search. It is common to require monotonicity from propagators ($\forall d_1, d_2 \in Dom. d_1 \subseteq d_2 \implies p(d_1) \subseteq p(d_2)$), but this does not model actual propagators well, since it excludes many types of random and heuristic propagators. If all propagators are monotonic, then the fix-point of propagators is unique, regardless of the order propagators are run [33, 55].

In practice, a single constraint may be implemented by a set of propagators, such as n^2 not equals propagators for an `all_different` constraint. We will leave this generalization out of the formalization, but note that it is straightforward.

Given two propagators p_1 and p_2 for a constraint c, p_1 is *stronger* than p_2 iff for all domains d, $p_1(d) \subseteq p_2(d)$, and for some domain d', $p_1(d') \subset p_2(d')$. A *consistency level* defines a specific strength of propagation. The strongest consistency level possible without violating the requirements for a propagator is *domain consistency* (also called (generalized) arc consistency, or complete propagation), where a propagator p removes all values for variables that have no supporting assignment in the associated constraint (formally, $\forall d \in Dom. p(d) = dom(asn(d) \cap c_p)$). There are other consistency levels defined in the literature, for example *value consistency* (also called forward checking), and *bound consistency*.

2.3 Constraint programming systems

Constraint programming systems are designed to enable the specification and solving of constraint models. Typical examples include open source solvers such as Gecode [18], Choco [45], and OR Tools [21] and commercial solvers such as SICStus Prolog [7] and CP Optimizer [27].

Constraint programming systems contain implementations for

Variables Variables can be Booleans, integers, floats, sets, and so on.

Propagators Propagators are the implementations of constraints. Systems typically provide many different propagators, for many different constraints.

[4] Except *local*, as their constraints and propagators are defined over all variables.

Branching A branching is an implementation of a heuristic, that decides how to make guesses in a search tree.

Search Search is used to find solutions to models comprised of variables and propagators combined with branchings. Search methods can be complete (DFS, Limited Discrepancy Search) or incomplete (Restart based search, LNS), and can be for satisfaction only or for optimizing..

3 Half-checking propagators

A *half-checking* propagator[5] is similar to a traditional propagator, only with less restrictions: if a solution is detected, then it is correct. In particular, half-checking propagators are allowed to actually *remove solutions*. Formally, a half-checking propagator is a function from domains to domains, with the properties that it is *local* and *contracting*, in addition to the following property:

Definition 2 (Half-checking). *The propagator p is* half-checking *for c, if for all assignments $a \in Asn$, if $p(dom(\{a\})) = dom(\{a\})$ then $a \in c$.*

Half-checking is a natural weakening of *checking*, where instead of requiring that all solutions to a constraint are precisely identified and thus the only fix-points of the function, we only require that fix-points of assignments must be solutions to the constraint. Also importantly, a half-checking propagator is not required to be *weakly monotonic* either. Since weak monotonicity is required for correctness, a half-checking propagator may actually be *incorrect*: it may remove an assignment that it would recognize as a solution.

Example 1. The *fail* propagator $\lambda d.\bot$ is a half-checking propagator for all constraints $c \in Con$. Since *fail* has no fix-points for any assignment, it is trivially half-checking. It is naturally contracting, as well as local, since all empty/failed domains are equated. The *fail* propagator is the strongest propagator possible, since $\forall d \in Dom. \bot \subseteq d$. It is also a rather useless propagator in practice, since it guarantees that no solution will be found.

Example 2. A propagator for $x < y$ is a half-checking propagator for $x \leq y$. This is easy to verify, since all solutions to the first are also solutions to the second.

Theorem 1. *All propagators are also half-checking propagators.*

Proof. This follows directly since *half-checking* is a weakening of *checking*. □

Theorem 2. *Solving a constraint model with half-checking propagators using solve is* sound*.*

Proof. All returned solutions from *solve* must be fix-points for all the propagators (by definition, whether traditional or half-checking). Since the only fix-points of both traditional and half-checking propagators are solutions to the associated constraint, the returned assignments are solutions to the model. □

[5] The name is inspired by the name half-reification. [14]

Theorem 3. *Solving a constraint model with half-checking propagators using solve is* not complete*.*

Proof. Given a model $\langle d, P\rangle$ with at least one solution. We can replace any propagator p in P with *fail* from Example 1 as a half-checking propagator for the constraint c_p. With *fail* in the set of propagators, no solutions are produced since there are no assignment fix-points for *fail*. □

4 Integrating half-checking propagators into a system

After defining and describing the theoretical properties of half-checking propagators, it is important to investigate how they can be supported in constraint programming systems. In most constraint programming systems, propagators are just components that interact with the current variables, and based on deductions may remove some values from its variables domains.

When implementing a propagator in a typical constraint programming system, the properties *contracting* and *local* are natural consequences of the programming interface: propagators only have access to their variables, and the only modifications that a propagator can do are removal of values from domains.

As shown in [49], if a constraint programming system uses re-computation [48, 18, 42] it may need to make adjustments for weakly-monotonic propagators as opposed to monotonic propagators. The reason is that running propagation twice may not give the exact same result, since the fix-point is no longer unique [33, 55]. Typical examples of this might be propagators that use randomized algorithms. The same situation naturally applies for half-checking propagators, and thus if the system is set up such that it can handle weakly-monotonic propagators, it can also handle half-checking propagators.

In addition to supporting half-checking propagators, there are additional practical concerns that need to be taken into account. When applicable, we will describe how this is done for the Gecode system.

4.1 Portfolio-based search

Half-checking propagators naturally leads to a incomplete search. In many cases this may be ok, but sometimes a user would like to know that all solutions have been found, that no solution exists, or that the optimal solution has been found. Using a cooperative portfolio solver combining an incomplete search with a complete search solves this, such as in the Failure Directed Search [56] used in the CP Optimizer [27] system, as is explored in [16] for scheduling problems where portfolios with some incomplete assets are used.

It is important to indicate to the portfolio system used that the asset with half-checking propagators is not a complete search method. If it is not possible to inform the system that an asset is incomplete, the resulting combined search may wrongly indicate that it is complete. In Gecode, returning false from the function called to set up the asset indicates that the asset is incomplete.

Given several half-checking propagators for constraints in a model, there are three main ways in which they can be used together in a portfolio system.

Combined All half-checking propagators can be combined in one asset.
Multiple assets For each half-checking propagator, create an asset in the portfolio that runs the problem with it. This may require too many assets.
Round robin A single asset can be used with a round-robin schedule that upon re-start switches between the different half-checking propagators to use.

Which strategy to use will depend on the problem at hand, the half-checking propagators, and the instances to solve. For any particular problem, it will require experimentation combined with experience in the behaviour of the half-checking propagators in question.

4.2 No-good recording

A crucial aspect for modern re-starting search is to record no-goods [32, 36]. A no-good is a constraint that describes the search-tree that has been explored so far, and is added upon re-start. In constraint programming, no-goods are typically based on negating the conjunction of a set of branching decisions. When combined with traditional constraint propagation for monotonic propagators, branching decisions precisely describe the explored part of a search tree. For weakly-monotonic propagators, the search-tree may not be precisely described by the no-good, but it is still correct.

In the presence of half-checking propagators, the parts of a search-tree that have been visited may contain solutions that were removed. Thus, a no-good from a search using half-checking propagators *is not globally valid*. It is still useful in the search using that half-checking propagator, but if it is used in an asset that claims to be complete, this will no longer be true.

Consider again the *fail* propagator from Example 1. Given a portfolio search with one asset a traditional and complete search, and one asset using *fail*. As soon as the latter is run it will fail and be done. Recording the no-good and posting it in the traditional asset will abort the search since the no-good would rule out the whole search tree.

4.3 Lazy clause generation

In lazy clause generation solvers [41], a propagator *explains* its deductions using clauses. There is nothing inherently problematic about combining half-checking propagators and lazy clause generation. One interesting aspect, is that a simple half-checking propagator that does some very mild extra propagation may produce clauses that are later on used in the no-good explanation clauses generated on failure, and may thus end up being used in a wider context.

For some half-checking propagators, such as the removal of crossing edges described in Section 6, generating good explanations is easy. For others, such as the approximation based upper bound computation in Section 7, useful explanations can be generated if the approximation produces a witness solution.

However, for some half-checking propagators such as the heuristic based filtering in Section 8, meaningful explanations may be quite hard to produce.

4.4 Testing of propagators

Propagators are complicated pieces of code, and propagator-specific testing is naturally needed to increase the confidence that a constraint programming system produces the correct results in addition to standard testing of algorithms. Unfortunately, half-checking propagators make the job of testing harder, since there are fewer guarantees that we can rely on.

Testing in the Gecode system is based on combining initial domains with a constraint checker for assignments used as an oracle. A constraint checker is typically a much simpler piece of code to write than the propagator under test. For all assignments in the initial domains, the testing system then removes values towards that assignment, running the propagator under test intermittently. If the assignment is in the constraint/validated by the check, the propagator should not remove the assignment, and otherwise the search should eventually fail. The whole idea relies on weak monotonicity, which half-checking propagators do not have. In addition, propagators may opt-in for extended checking of bounds and domain consistency, neither of which are useful to a half-checking propagator.

In [1] metamorphic testing is used to test constraint propagators. The idea is to use an extensional constraint with a table propagator as a validation propagator. A test consists of running original propagator and the validation propagator, and then comparing the resulting search trees. Again, the fact that a propagator must be weakly monotonic and checking are crucial properties here.

A similar idea is explored in SolverCheck [19]: initial domains and a constraint checker are used to generate a list of valid assignments. These assignments are then used to build reference propagators, including weakening them to build bounds-consistent propagators. Propagation of the propagator under test is compared with the simple reference propagator. Again the assumption is naturally that propagators are correct, and will not remove solutions.

Since half-checking propagators are allowed to remove solutions, none of the above testing strategies will work. However, there are some things that we could test for, namely the half-checking property. For example, using the Gecode testing strategy it is possible to adjust it to only check that a solution accepted by the propagator was also verified by the checker as being valid.

Some half-checking propagators use reasoning that is only valid for optimal assignments for the constraint (e.g., the propagators in Sections 6 and 7). For such propagators, the Gecode testing framework can be used as-is specifically for optimal assignments.

5 The `cost-circuit` constraint and TSP

In the following three sections, examples of general techniques and strategies to use when implementing half-checking propagators are given. For each one, an algorithm is proposed for the `cost-circuit` constraint. This section describes the constraint and the Travelling Salesperson Problem that it is used for. The three half-checking propagators introduced are evaluated in Section 9.

5.1 Theory

Let $G = \langle V, E \rangle$ be a graph consisting of a set of vertices or nodes V and a set of edges $E \subseteq V \times V$ indicating which edges are connected. The graph is *complete* if $E = V \times V$, i.e., all nodes are connected to all other nodes. The graph may be *directed* or *undirected.* A *path* of length k in a graph is a sequence of nodes $\langle v_1, v_2, \dots v_k \rangle$ where $\forall_{i \in 1 \dots k-1}. \langle v_i, v_{i+1} \rangle \in E$. A path is a *circuit* when $\langle v_k, v_1 \rangle \in E$. When all nodes are unique it is called a *simple path* and a *cycle* or a *simple circuit.* When a simple path or a simple circuit covers all the nodes ($k = |V|$), it is called Hamiltonian, and finding such are one of the classical NP-complete problems [31]. A graph is *connected* when there exists a path between all pairs of nodes. A *tree* is a graph that is connected and has no cycles. A *weight function* w is a function from edges to real numbers ($w \in E \to \mathbb{R}$), and most often to non-negative real numbers. It is *symmetric* if $\forall_{v_1, v_2 \in V} w(\langle v_1, v_2 \rangle) = w(\langle v_2, v_1 \rangle)$. A weight function *respects the triangle inequality* when $\forall_{v_1, v_2, v_3 \in V} w(\langle v_1, v_3 \rangle) \leq w(\langle v_1, v_2 \rangle) + w(\langle v_2, v_3 \rangle)$. Given a graph $G = \langle V, E \rangle$ and a weight function w, a *minimum spanning tree* (MST) $M = \langle V, T \rangle$ is a tree with the same nodes as the graph, with $T \subseteq E$, and with a minimum weight.

The *Travelling Salesperson Problem* (TSP) is the problem of given a graph $G = \langle V, E \rangle$ and a weight function w, find a Hamiltonian circuit for the graph with minimum weight. This is the natural weighted extension of the Hamiltonian circuit problem. It is common to require that the graph for a TSP is complete; a missing edge can be modelled as an arbitrary large weight. If the nodes of the graph have positions and the weight is defined as the distance between the nodes, it is a *Euclidean* TSP. The TSPLIB [46] is a collection of 110 real-world TSP instances, with 77 using Euclidean 2D-distance.

5.2 TSP in constraint programming

The `circuit`(S) constraint models the Hamiltonian circuit problem using an array of successor variables S, where $S_i = j$ indicates that j is the successor of i in the circuit. The `cost-circuit`(S, w, c) is the same, with the variable c representing the total cost of the circuit according to the weight function w.

The `circuit` constraint is one of the classical global constraints in constraint programming [35, 3]. Since the base problem is NP-complete, filtering algorithms are focused on effective but not complete filtering. The base filtering is handled by the embedded implied `all_different`(S), with additional removal of edges that would lead to circuits smaller than $|S|$ (subtour elimination). In addition, many other structural filters have been identified and propagated (e.g., [49, 17]). For the weighted variant, there have been recent advances above the basic filtering, for example in [4] and [28].

The above propagation algorithms are all limited by the fact that no correct value may be removed. State of the art TSP solvers such as Concorde [9] can do more, since the goal is to find a single optimal solution, not all possible solutions.

In constraint programming, the choice of the branching heuristic is key. For TSP, several different heuristics have been proposed [13, 28], with no clear winner. Here, we will focus on the *Warnsdorff* heuristic [57] for the Knights tour problem (and more generally, the Hamiltonian path problem). The heuristic is, when cast in constraint programming terms, comprised of two parts. The first is the variable ordering: assigning variables along a path that is built up incrementally. The second is the value ordering: preferring to go to nodes with the lowest out-degree. Adjusted for the case of complete graphs with distances, the out-degree is less important and using the minimum distance becomes more important.

6 Technique: Dominating solutions

When solving a constraint programming problem it is common to see that one solution may *dominate* another solution [23], either because of symmetries or because of one solution having better cost. Propagation for symmetries is common [15], as is more global views for symmetry breaking [39]. For cost-dominating solutions, there is less opportunities for incorporating the domination relation into propagators, since it is typically quite specialized and will not behave as a traditional propagator. This is a clear opportunity to apply half-checking propagators.

6.1 No Crossing Lines

In a pure Euclidean TSP over a complete graph with no side-constraints, a property that always holds is that in a optimal solution there are no crossing lines: given two crossing lines $\langle s_1, e_1\rangle$ and $\langle s_2, e_2\rangle$, they can be replaced with $\langle s_1, e_2\rangle$ and $\langle s_2, e_1\rangle$, which will have the same or lower weight. Thus, any solution that contains crossing lines will be dominated by a solution in which the crossing lines are un-crossed. For an edge e, let $cl(e) \subset E$ be the set of lines that cross it.

Using this observation, we can design our first interesting half-checking propagators, which we call `ncl`(S) for No Crossing Lines. The key observation is that given an assignment that includes an edge e in the solution, we known that in no *optimal* solution where e is used (if any such exist), are any of the lines in $cl(e)$ used. Note that there may be no optimal solution including the edge e. Given an assignment $S_i = j$, for all edges $\langle k, l\rangle \in cl(\langle S_i, S_j\rangle)$, propagate $S_k \neq l$.

For a solution that uses Warnsdorff's rule for variable selection, it is possible to choose a simpler filtering called `ncl-path`(S, f). The propagator follows the Warnsdorff path from the starting node f to the last known node in the path, and removes any outgoing edges from that node that cross the fixed path. To avoid unnecessary failures, the propagator avoids pruning if it would directly cause a failure (similar to the recent idea of non-failing propagators for LNS [5]).

Stronger reasoning using crossing lines is also possible. For a node i with domain d_{S_i}, any edge crossing all remaining outgoing edges from i can be removed ($\langle k, l\rangle \in \cap_{v \in d_{S_i}} cl(\langle S_i, v\rangle)$). We have not implemented this stronger propagation.

Implementation. Implementing `ncl` requires a fast and efficient look-up of the *cl* sets. Since the graph is fixed, we pre-compute this information. In a complete TSP with n nodes, the number of edges is n^2, which means that the number of crossing lines is $O\left(n^4\right)$, a very large number for even a modest number of cities. Thus, the propagator can only be used for quite small instances.

For `ncl-path`, the crossing lines are computed on the fly instead. Along the Warnsdorff path, n assignments will be made, and for each assignment $O(n)$ other edges need to be considered. Thus, along a path a maximum of $O(n^2)$ pairs of edges are considered. This is much less taxing than the full `ncl` propagation.

To speed up the computation of the crossing lines, a spatial index is used to make geometric look-ups. Our index is based on the STR [37] construction of R-trees [24]. We adjusted it in two ways. The first is to make binary subdivisions recursively. The second is to first sort objects based on width/height, and then on position. This strategy is useful since very long edges that cover most other edges are pushed to one side of the tree. Using this ordering instead of the normal STR ordering gave a small but significant speed-up.

7 Technique: Heuristic bounds

For many hard problems in computer science, there are algorithms defined that create good but not provably optimal solutions. Such algorithms are often constructive, producing a witness solution for the bound.

Bounds are typically used in constraint programming propagators for the worst case, i.e., finding the lowest and the highest weight possible. The difference here is that we instead strive to give good and tight upper-bounds based on a best-effort to find a solution to a single constraint. Naturally, such bounds may be invalid in the presence of other constraints in the model, but if they are valid, they will help guide propagation. Note that filtering the upper bound for an optimization variable is not effective, since it is bound from above by the search.

7.1 Christofides-Serdyukov bounds propagation

The classical approximation algorithm for a metric TSP (such as Euclidean TSPs) is the Christofides-Serdyukov algorithm [8, 51]. The algorithm is defined for a complete graph $G = \langle V, E\rangle$ with Euclidean weights w. The idea is to find a minimum spanning tree of the graph, and augment it with a minimum weight matching among the nodes with odd degree in it. Given this graph, an Euler circuit skipping visited nodes represents the approximation and is at most 1.5 times the length of the optimal circuit. The Christofides-Serdyukov algorithm is very popular as a reasonably simple algorithm that gives a good bound[6]. For example, it is implemented as a stand-alone TSP solver in OR Tools [21].

We propose the $\texttt{cbp}(S, w, c)$ bounds propagator, that works as follows. Let $G_S = \langle V, E_S \rangle$ be the current graph induced by the S variables, with G the original graph. For simplicity, we treat the graph as undirected. Our algorithm proceeds as follows

- Find a spanning tree of G_S, M_S, with the fixed edges in S included.
- Let O be the set of edges with odd degree in M_S.
- Find a *maximal* matching in G_S for the edges in O, and add to M_S.
- For the nodes not matched in the previous step, find a matching using the edges in G and add to M_S.
- Construct an Euler circuit in M_S, and follow it, skipping any node that has been used before with the corresponding edge in E (even if it is not in E_S).
- Adjust the upper bound of c to be at most the weight of the found circuit.

The above algorithm tries as far as possible to use only edges in the graph G_S. If only such edges are used, then the upper bound represents a solution to the sub-problem. Otherwise, the best remaining tour may have a larger cost.

Implementation The implementation of the `cbp` propagators follows the outline above. The spanning tree is found using a variant of Kruskals algorithm [34][7] First all fixed edges are added to the tree, and then the edges in the graph are traversed in increasing order. For this, our graphs keep a list of all the edges in increasing weight order. If $|E_s| > \frac{1}{4}|E|$, then this list is used with a filter to check for validity, otherwise a new list is constructed from the current domains. The constant $\frac{1}{4}$ was determined through experimentation, and needs to be adjusted for a specific implementation. Finding the Euler walk is done using Hierholzers algorithm [26], with the stack-based formulation.

Instead of the complete minimum weight matching used in Christofides-Serdyukov, a simple greedy algorithm is used instead. This is because implementing and running a maximal matching algorithm such as Edmonds algorithm [12] is both complicated and time-consuming. An approximate solution here may give a higher bound, but never a wrong one.

8 Technique: Heuristic deductions

This is the most general technique, where heuristic algorithms are used to make inferences and deductions that may or may not be true.

[6] The $3/2$ approximation bound has been the best known since 1976, with a recent pre-print the first to give a better bound at $3/2 - \epsilon$ for some $\epsilon > 10^{36}$ [30].

[7] It is also be possible to adapt either Borůvka's algorithm [6, 40] by adding all fixed edges as the first step or the Jarník-Prim-Dijkstra algorithm [29, 44, 11, 40] by extending the growing tree with fixed edges when they become adjacent. For Euclidean TSPs, it is possible to find the MST in $O(|V| \log |V|)$ using the Delauney triangulation [10] as the graph [52] using one of the previously mentioned algorithms.

8.1 Heuristic 1-tree propagation

As discussed in [25, 4, 28], a 1-tree is a very useful structure for analysing properties of graphs when searching for weighted Hamiltonian circuits. Formally, a 1-tree for a graph $G = \langle VE \rangle$ and a node n_1 is a spanning tree for the graph $\langle V \setminus \{n_1\}, E \setminus \{\langle n, n' \rangle | n = n_1 \vee n' = n_1\} \rangle$ along with a set of two edges from n_1 to the rest of the graph: $\{\langle n, n_1 \rangle, \langle n_1, n' \rangle\}$. A minimum 1-tree is a 1-tree with minimum weight. A circuit is a 1-tree for any choice of node n_1 in the graph.

Our `one-tree` propagator starts by choosing a node n_1 to use as the dedicated node, after which a 1-tree is computed. Three rules are used: Update the lower bound of the cost with the cost of the 1-tree; If the 1-tree is a circuit, set this as the solution; For some node with degree > 2 in the spanning tree part, remove the longest of the incident edges. The latter idea is inspired by Held and Karps [25] techniques from MIP formulations of the TSP problem, where the residual costs of the edges in such nodes are manipulated iteratively.

The choice of n_1 determines the bound. As such, the node with the largest sum of its two smallest incident edges gives the strongest bounds propagation.

Implementation. To find a 1-tree, the implementation uses an algorithm based on Kruskals algorithm similar to the implementation of the spanning tree algorithm in 7.1. The main difference is that the special node n_1 is given as an additional argument, and the algorithm returns a spanning tree for $V \setminus \{n_1\}$, and two edges incident to n_1. First all fixed edges are added, either to the spanning tree or to the n_1 edges. While processing edges to build up the spanning tree, if an edge is incident to n_1 add it to that set unless it already contains 2 edges. When the spanning tree is constructed, we may still not have 2 edges in the n_1 set, and if so add the smallest. Note that the algorithm is only executed after normal propagation for the circuit constraint has been done. Thus, we can assume that there are at most 2 fixed edges incident to n_1.

9 Evaluation

Our implementation[8] is done using the Gecode [18] constraint programming system, version 6.2.0. The main constraint in the model is `cost-circuit`, along with an `inverse` constraint to get variables representing the predecessors also. The main branching heuristic used is the Warnsdorff heuristic for selecting the variable to branch on with randomized starting nodes, and for values selecting the value with min weight (slightly randomized, inspired by [2]). Instances are read from TSPLIB files. Our experiments are run on a a Macbook Pro 15 with a 6-core 2.7 GHz Intel Core i7 processor and 16 GiB memory. The experiments are not for deciding the best way to solve a TSP using constraint programming, the aim is to demonstrate that the techniques adds filtering.

Computing the crossing lines data-structure from Section 6.1 quickly starts to get costly. At around 50 nodes, it takes 0.25-0.3 seconds and at around 100

[8] Available at `https://github.com/zayenz/half-checking-propagators`

	`ncl-path`			`cbp`			`one-tree`			All		
Instance	S	min	max	S	min	max	S	min	max	S	min	max
berlin52	99.54%	=	=	=	=	15.83%	99.95%	88.02%	=	99.49%	88.02%	15.83%
st70	99.87%	=	=	=	=	13.04%	99.97%	79.89%	=	99.85%	79.89%	13.04%
eil51	99.61%	=	=	=	=	17.39%	99.95%	90.57%	=	99.57%	90.57%	17.39%
eil76	=	=	=	=	=	14.82%	99.98%	93.84%	=	99.98%	93.84%	14.82%
eil101	99.65%	=	=	=	=	11.72%	99.99%	93.78%	=	99.63%	93.78%	11.72%
lin105	99.84%	=	=	=	=	7.30%	99.99%	61.94%	=	99.83%	61.94%	7.30%
lin318	=	=	=	=	=	4.87%	=	66.32%	=	=	66.32%	4.87%
pr76	99.89%	=	=	=	=	10.74%	=	76.25%	=	99.89%	76.25%	10.74%
pr107	=	=	=	=	=	5.83%	=	63.71%	=	=	63.71%	5.83%
pr124	99.58%	=	=	=	=	5.80%	=	73.48%	=	99.58%	73.48%	5.80%
pr136	99.99%	=	=	=	=	8.08%	⊥	⊥	⊥	⊥	⊥	⊥
pr144	=	=	=	=	=	5.50%	=	42.42%	=	=	42.42%	5.50%
pr152	99.40%	=	99.49%	=	=	4.71%	=	58.93%	=	99.40%	58.93%	4.71%

Table 1. Propagator filtering strength. Reported is the reduction when using the propagators `ncl-path`, `cbp`, `one-tree`, and all combined on the domains size of S and the min and max cost after assigning 10%. = means no reduction, $\perp$ means a failure.

nodes it takes 0.8-1.1 seconds. However, for `lin318` with 318 nodes, it takes more than 5 minutes to compute, which is clearly too long to be useful. In the following, we will skip the full version since it is clearly impractical.

Table 1 reports the filtering improvements for our proposed propagators. Five variants are run simultaneously, assigning 10% of the nodes in the path. The variants use the standard model, along with variants with the propagators `ncl-path`, `cbp`, `one-tree`, and all three combined. The reported value is the reduction in the sum domain size of the successor variables S, and the adjustment of the minimum and maximum costs compared with the standard model. As can be seen, our propagators have complementary and strong filtering.

Finding good solutions quickly is naturally desired. Unfortunately, the improved filtering does not translate into better solving directly. For some test-cases, our propagators give modestly better results for solving under time-limits. However, we believe that a main issue is that it is not possible yet to generate no-goods local to an asset in Gecode. Further investigation is clearly needed, as is testing other problems using the `cost-circuit` constraint.

10 Related work

The requirement of correctness for propagators have been a constant in constraint programming since the field began. Still, there are a few techniques and approaches that have touched on similar ideas.

The most similar technique to half-checking propagators is probably *streamlining constraints* [20]. The idea is to post additional constraints in a model in order to focus on certain subsets of solutions that exhibit some kinds of regularities. Typically, these are found examining solutions to small instances, and the added streamliners help find these regularities in larger instances. The idea is similar to half-checking propagators, in that in order to solve a problem we may want to rule out potential solutions. In a certain sense, the `ncl` propagator is a streamliner constraint, since we focus on the solutions that have the no-crossing lines regularity. On the other hand, the `cbp` and `one-tree` propagators we propose are not easily formulated as streamliners. An additional difference is that half-checking propagators focus on adding new reasoning for existing constraints, while streamliner constraints focus on adding new reasoning for models.

The similar approaches of cost propagation [22] and belief propagation [43] use a domain store that indicates a common cost or belief for each variable-value pair. Both approaches use the gathered information to guide the search (a non-backtracking search for [22]). As remarked by Pesant in [43] a value that gets a belief very close to 0 (or perhaps even 0, due to rounding errors), is very unlikely to be in any solution, and thus it might be beneficial to actually remove these values. Such a filtering rule would be a half-checking propagator.

In [4], TSP instances tested are pre-processed with tight bounds based on standard state-of-the-art heuristics. While it is not clearly stated, this pre-processing is of course not valid if there are any other constraints in the instances than just a `cost-circuit`. This kind of bounds updates is similar to what we propose in Section 7, although we use it continuously during search.

In [50], Sellmann and Harvey propose using heuristic constraint propagation. While it may sound similar to half-checking propagators and especially the techniques we present, the crucial difference it that Sellmann and Harvey focus on incomplete, but still *correct* propagation.

11 Conclusions

This paper has introduced *half-checking propagators*, a new variant of propagators that are not required to be correct. Lifting this restriction opens up new possibilities for designing propagation algorithms. The goal is to guide search towards good solutions. To regain completeness, we paired models with half-checking propagators in a portfolio with standard models.

A detailed description on how to integrate half-checking propagators into modern constraint programming systems was given. To showcase the idea, three techniques for designing half-checking propagators were presented and made concrete with an application to the `cost-circuit` constraint.

The most important future work is of course to make computational studies on how to best use half-checking propagators. In order to make this as fair as possible, an improvement to Gecode that would allow us to record no-goods locally in assets with half-checking propagators is needed. There are many examples of hard problems, where half-checking propagators could be useful, with the problems tested in [5] an interesting list to start with.

We think that scheduling problems may be an interesting future area of research for new half-checking propagators. Also, studying automatically generated streamliner constraints [58, 54] could be an interesting source of ideas for new half-checking propagators.

Acknowledgments Thanks to the anonymous reviewers of this paper, which helped improve and clarify many points and recommended additional references.

References

1. Akgün, Ö., Gent, I.P., Jefferson, C., Miguel, I., Nightingale, P.: Metamorphic testing of constraint solvers. In: Hooker, J. (ed.) Principles and Practice of Constraint Programming. pp. 727–736. Springer International Publishing, Cham (2018)
2. Archibald, B., Dunlop, F., Hoffmann, R., McCreesh, C., Prosser, P., Trimble, J.: Sequential and parallel solution-biased search for subgraph algorithms. In: Rousseau, L., Stergiou, K. (eds.) Integration of Constraint Programming, Artificial Intelligence, and Operations Research - 16th International Conference, CPAIOR 2019, Thessaloniki, Greece, June 4-7, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11494, pp. 20–38. Springer (2019). https://doi.org/10.1007/978-3-030-19212-9_2
3. Beldiceanu, N., Contejean, E.: Introducing global constraints in CHIP. Mathematical and Computer Modelling **20**(12), 97 – 123 (1994). https://doi.org/https://doi.org/10.1016/0895-7177(94)90127-9
4. Benchimol, P., van Hoeve, W.J., Régin, J., Rousseau, L., Rueher, M.: Improved filtering for weighted circuit constraints. Constraints **17**(3), 205–233 (2012). https://doi.org/10.1007/s10601-012-9119-x
5. Björdal, G., Flener, P., Pearson, J., Stuckey, P.J., Tack, G.: Solving satisfaction problems using large-neighborhood search. In: Simonis, H. (ed.) Principles and Practice of Constraint Programming – CP 2020 (2020)
6. Borůvka, O.: O jistém problému minimálním. Práce Moravské přírodovědecké společnosti, Mor. přírodovědecká společnost (1926)
7. Carlsson, M., Ottosson, G., Carlson, B.: An open-ended finite domain constraint solver. In: Glaser, H., Hartel, P., Kuchen, H. (eds.) Programming Languages: Implementations, Logics, and Programs. pp. 191–206. Springer Berlin Heidelberg, Berlin, Heidelberg (1997)
8. Christofides, N.: Worst-case analysis of a new heuristic for the traveling salesman problem. Tech. Rep. 388, Graduate School of Industrial Administration, Carnegie Mellon University (1976)
9. Cook, W.: Concorde TSP Solver, `http://www.math.uwaterloo.ca/tsp/concorde.html`
10. Delauney, B.: Sur la sphìre vide. A la mémoire de Georges Voronoï. Izvestia Akademia Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk **7**, 793–800 (1934)
11. Dijkstra, E.W.: A note on two problems in connexion with graphs. Numerische mathematik **1**(1), 269–271 (1959)
12. Edmonds, J.: Paths, trees, and flowers. Canadian Journal of Mathematics **17**, 449–467 (1965)
13. Fages, J.G., Lorca, X., Rousseau, L.M.: The salesman and the tree: the importance of search in CP. Constraints **21**(2), 145–162 (Apr 2016). https://doi.org/10.1007/s10601-014-9178-2
14. Feydy, T., Somogyi, Z., Stuckey, P.J.: Half reification and flattening. In: Lee, J. (ed.) Principles and Practice of Constraint Programming – CP 2011. pp. 286–301. Springer Berlin Heidelberg, Berlin, Heidelberg (2011)
15. Flener, P., Frisch, A.M., Hnich, B., Kiziltan, Z., Miguel, I., Pearson, J., Walsh, T.: Breaking row and column symmetries in matrix models. In: Principles and Practice of Constraint Programming - CP 2002, 8th International Conference, CP 2002, Ithaca, NY, USA, September 9-13, 2002, Proceedings. pp. 462–476 (2002). https://doi.org/10.1007/3-540-46135-3_31

16. Fontaine, D., Michel, L., Van Hentenryck, P.: Parallel composition of scheduling solvers. In: Quimper, C.G. (ed.) Integration of AI and OR Techniques in Constraint Programming. pp. 159–169. Springer International Publishing, Cham (2016)
17. Francis, K.G., Stuckey, P.J.: Explaining circuit propagation. Constraints **19**(1), 1–29 (Jan 2014). https://doi.org/10.1007/s10601-013-9148-0
18. Gecode team: Gecode, the generic constraint development environment (2018), `http://www.gecode.org/`
19. Gillard, X., Schaus, P., Deville, Y.: SolverCheck: Declarative testing of constraints. In: Schiex and de Givry [47], pp. 565–582. https://doi.org/10.1007/978-3-030-30048-7_33
20. Gomes, C.P., Sellmann, M.: Streamlined constraint reasoning. In: Wallace, M. (ed.) Principles and Practice of Constraint Programming - CP 2004, 10th International Conference, CP 2004, Toronto, Canada, September 27 - October 1, 2004, Proceedings. Lecture Notes in Computer Science, vol. 3258, pp. 274–289. Springer (2004). https://doi.org/10.1007/978-3-540-30201-8_22
21. Google: OR-Tools (2019), `https://developers.google.com/optimization`
22. Grohe, B., Wedelin, D.: Cost propagation – numerical propagation for optimization problems. In: Perron, L., Trick, M.A. (eds.) Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems. pp. 97–111. Springer Berlin Heidelberg, Berlin, Heidelberg (2008)
23. Guns, T., Stuckey, P.J., Tack, G.: Solution dominance over constraint satisfaction problems. In: Leo, K. (ed.) The 17th workshop on Constraint Modelling and Reformulation (2018), `http://arxiv.org/abs/1812.09207`
24. Guttman, A.: R-trees: A dynamic index structure for spatial searching. SIGMOD Rec. **14**(2), 47–57 (Jun 1984). https://doi.org/10.1145/971697.602266, `https://doi.org/10.1145/971697.602266`
25. Held, M., Karp, R.M.: The traveling-salesman problem and minimum spanning trees: Part ii. Mathematical Programming **1**(1), 6–25 (Dec 1971). https://doi.org/10.1007/BF01584070
26. Hierholzer, C., Wiener, C.: Ueber die möglichkeit, einen linienzug ohne wiederholung und ohne unterbrechung zu umfahren. Mathematische Annalen **6**(1), 30–32 (1873)
27. IBM: IBM ILOG CP Optimizer (2019), `https://www.ibm.com/analytics/cplex-cp-optimizer`
28. Isoart, N., Régin, J.: Integration of structural constraints into TSP models. In: Schiex and de Givry [47], pp. 284–299. https://doi.org/10.1007/978-3-030-30048-7_17
29. Jarník, V.: O jistém problému minimálním. Práca Moravské Prírodovedecké Spolecnosti **6**, 57–63 (1930)
30. Karlin, A.R., Klein, N., Gharan, S.O.: A (slightly) improved approximation algorithm for metric TSP (2020), `https://arxiv.org/abs/2007.01409`
31. Karp, R.M.: Reducibility among Combinatorial Problems, pp. 85–103. Springer US, Boston, MA (1972). https://doi.org/10.1007/978-1-4684-2001-2_9
32. Katsirelos, G., Bacchus, F.: Generalized nogoods in CSPs. In: Veloso, M.M., Kambhampati, S. (eds.) Proceedings, The Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference, July 9-13, 2005, Pittsburgh, Pennsylvania, USA. pp. 390–396. AAAI Press / The MIT Press (2005), `http://www.aaai.org/Library/AAAI/2005/aaai05-062.php`
33. Knaster, B.: Un théorème sur les fonctions d'ensembles. Annales Soc. Polonaise **6**, 133—134 (1928)

34. Kruskal, J.: On the shortest spanning subtree of a graph and the traveling salesman problem. Proceedings of the American Mathematical Society **7**(1), 48–50 (1956)
35. Lauriere, J.L.: A language and a program for stating and solving combinatorial problems. Artificial Intelligence **10**(1), 29 – 127 (1978). https://doi.org/https://doi.org/10.1016/0004-3702(78)90029-2
36. Lee, J.H.M., Schulte, C., Zhu, Z.: Increasing nogoods in restart-based search. In: Schuurmans, D., Wellman, M. (eds.) AAAI Conference on Artificial Intelligence. pp. 3426–3433. AAAI Press, Phoenix, AZ, USA (Feb 2016), `https://chschulte.github.io/papers/leeschulteea-aaai-2016.html`
37. Leutenegger, S., Lopez, M., Edgington, J.: STR: A simple and efficient algorithm for r-tree packing. Proc. VLDB Conf pp. 497–506 (05 1997). https://doi.org/10.1109/ICDE.1997.582015
38. Mackworth, A.: Consistency in networks of relations. Artificial Intelligence **8**(1), 99–118 (1977)
39. Mears, C., de la Banda, M.G., Demoen, B., Wallace, M.: Lightweight dynamic symmetry breaking. Constraints **19**(3), 195–242 (2014). https://doi.org/10.1007/s10601-013-9154-2
40. Nešetril, J., Nešetrilová, H.: The origins of minimal spanning tree algorithms–Bůruvka and Jarnık. Documenta Mathematica pp. 127–141 (2012)
41. Ohrimenko, O., Stuckey, P., Codish, M.: Propagation = lazy clause generation. In: Bessiere, C. (ed.) Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming,. LNCS, vol. 4741, pp. 544–558 (2007)
42. Perron, L.: Search procedures and parallelism in constraint programming. In: Jaffar, J. (ed.) Principles and Practice of Constraint Programming – CP'99. pp. 346–360. Springer Berlin Heidelberg, Berlin, Heidelberg (1999)
43. Pesant, G.: From support propagation to belief propagation in constraint programming. J. Artif. Intell. Res. **66**, 123–150 (2019). https://doi.org/10.1613/jair.1.11487
44. Prim, R.C.: Shortest connection networks and some generalizations. The Bell System Technical Journal **36**(6), 1389–1401 (1957)
45. Prud'homme, C., Fages, J.G., Lorca, X.: Choco Documentation. TASC - LS2N CNRS UMR 6241, COSLING S.A.S. (2017), `http://www.choco-solver.org`
46. Reinelt, G.: TSPLIB - A Traveling Salesman Problem Library. INFORMS Journal on Computing **3**(4), 376–384 (1991), `http://dblp.uni-trier.de/db/journals/informs/informs3.html#Reinelt91`
47. Schiex, T., de Givry, S. (eds.): Principles and Practice of Constraint Programming - 25th International Conference, CP 2019, Stamford, CT, USA, September 30 - October 4, 2019, Proceedings, Lecture Notes in Computer Science, vol. 11802. Springer (2019). https://doi.org/10.1007/978-3-030-30048-7
48. Schulte, C.: Programming Constraint Services, Lecture Notes in Artificial Intelligence, vol. 2302. Springer-Verlag (2002), `https://link.springer.com/book/10.1007/3-540-45945-6`
49. Schulte, C., Tack, G.: Weakly monotonic propagators. In: Gent, I. (ed.) Fifteenth International Conference on Principles and Practice of Constraint Programming. Lecture Notes in Computer Science, vol. 5732, pp. 723–730. Springer-Verlag, Lisbon, Portugal (Sep 2009). https://doi.org/10.1007/978-3-642-04244-7_56
50. Sellmann, M., Harvey, W.: Heuristic constraint propagation. In: Hentenryck, P.V. (ed.) Eighth International Conference on Principles and Practice of Constraint Programming. Lecture Notes in Computer Science, vol. 2470, pp. 738–743. Springer-Verlag, Ithaca, NY, USA (Sep 2002). https://doi.org/https://doi.org/10.1007/3-540-46135-3_55

51. Serdyukov, A.I.: O nekotorykh ekstremal'nykh obkhodakh v grafakh (on some extremal walks in graphs). Upravlyaemye sistemy **17**, 76–79 (1976)
52. Shamos, M.I., Hoey, D.: Closest-point problems. In: Proceedings of the 16th Annual Symposium on Foundations of Computer Science. p. 151–162. SFCS '75, IEEE Computer Society, USA (1975). https://doi.org/10.1109/SFCS.1975.8
53. Shaw, P.: Using constraint programming and local search methods to solve vehicle routing problems. In: Maher, M., Puget, J.F. (eds.) Fourth International Conference on Principles and Practice of Constraint Programming. Lecture Notes in Computer Science, vol. 1520, pp. 417–431. Springer-Verlag, Pisa, Italy (1998)
54. Spracklen, P., Dang, N., Akgün, Ö., Miguel, I.: Automatic streamlining for constrained optimisation. In: Schiex, T., de Givry, S. (eds.) Principles and Practice of Constraint Programming - 25th International Conference, CP 2019, Stamford, CT, USA, September 30 - October 4, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11802, pp. 366–383. Springer (2019). https://doi.org/10.1007/978-3-030-30048-7_22
55. Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific Journal of Mathematics **5**(2), 285–309 (1955), `https://projecteuclid.org:443/euclid.pjm/1103044538`
56. Vilím, P., Laborie, P., Shaw, P.: Failure-directed search for constraint-based scheduling. In: CPAIOR '15: Proceedings of the 12th International Conference on Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems. Springer-Verlag (2015)
57. von Warnsdorff, H.C.: Des rösselsprungs einfachste und allgemeinste lösung. Th. G. Fr. Varnhagensehen Buchhandlung (1823)
58. Wetter, J., Akgün, Ö., Miguel, I.: Automatically generating streamlined constraint models with Essence and Conjure. In: Pesant, G. (ed.) Principles and Practice of Constraint Programming - 21st International Conference, CP 2015, Cork, Ireland, August 31 - September 4, 2015, Proceedings. Lecture Notes in Computer Science, vol. 9255, pp. 480–496. Springer (2015). https://doi.org/10.1007/978-3-319-23219-5_34